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Generalized Wagner model for 2D symmetric and elastic bodies.

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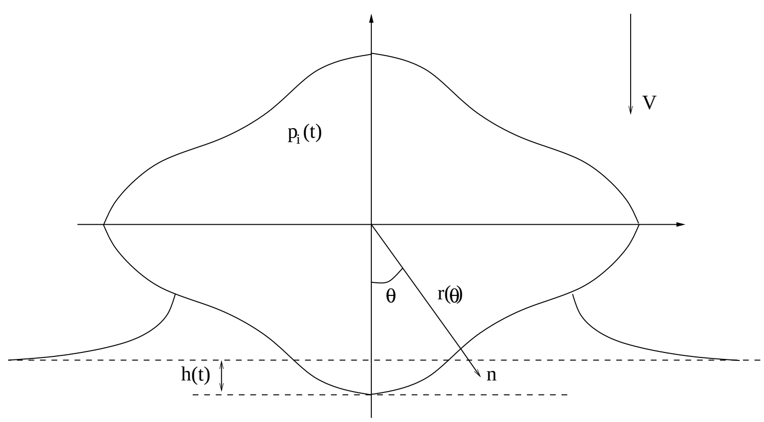
1) Introduction

We propose an algorithm to solve the generalized Wagner problem for two dimensional symmetric and elastic bodies. Following Zhao *et al.* (1996), the word generalized means that the boundary condition on the wetted surface is prescribed on its exact position. On the other hand, the boundary condition on the free surface is linearized on lines emanating from the contact points. This problem does not pose difficulties anymore for a rigid body whatever its shape, symmetric or not (see Malleron *et al.* (2007)). The main difficulty we are facing now is to couple the hydrodynamic problem with the elastic deformations of the body. In practice, we must deal with the time varying shape of the wetted surface. This makes the problem highly nonlinear since we want to solve a fully coupled problem. That is to say we prescribe the continuity of both the stress and the velocity at the wetted interface. The usual way to solve the rigid case is to decompose the inverse velocity of expansion of the wetted surface as polynomials of the position of the contact point (see Mei *et al.* (1999) but also Wagner (1932)). This is quite reasonable for smooth body shape. Time hence becomes a parameter. Then the Wagner condition (continuity of the vertical displacement at the contact point) is solved by collocation thus providing the contact point at any instant since we previously determined its history. Fortunately, even if the deformations might be high the shape is always smooth (no cusp). Hence, we could solve the hydro-elastic problem in the same spirit. The time history of the elastic shape has to be evaluated. The aim of the present study is to show how to proceed that way.

2) Boundary value problem

The boundary value problem is illustrated on the figure below



It is formulated for the velocity potential φ and the deflection w.

$$\begin{cases} \Delta\varphi = 0 & y < 0 \\ \varphi = 0 & \text{on the free surface} \\ \varphi_{,n} = \vec{V}\cdot\vec{n} + w_{,t}(x,y,t) & \text{on the wetted surface } D(t) \\ \varphi \to 0 & (x^2+y^2) \to \infty \\ \dot{h} = V(t) & \\ L(w) + \rho_s e(\ddot{w} - \dot{V}) = p(x,y,t) - p_i(t) & \text{on the wetted part of the deformable body} \\ w(x,y,0) = h(0) = 0 & \text{at initial time } t = 0 \\ \dot{h} = V_{ini} & \text{at } t = 0 \\ M\dot{\vec{V}} = -\int_{D(t)} p(x,y,t)\vec{n}ds & \text{Newton law of free falling} \end{cases} \quad (1)$$

It corresponds to the free falling of a deformable body onto a liquid initially at rest. The simulation starts when the body hits the liquid. In the sequel the velocity of penetration $\vec{V}$ is directed downwards and it is noted $\vec{V} = -V\vec{y}$. The normal $\vec{n}$ is directed towards the fluid. The equations which couple the structural and hydrodynamic problems are the continuity conditions at the wetted surface, both written in terms of stress and normal velocity. The deflection of the membrane is measured, in a Lagrangian way, from its undeformed shape. The normal velocity due to the deformation is noted $\dot{w}$. On the other hand, the condition of stress continuity makes appear an operator $L(w)$ which is function of the considered shape. Actually it may contain nonlinear terms. However these terms are spatial derivatives of w only.

In a first attempt the structure is supposed to have no bending stiffness and to be non extensible (constant length of the total chord). Hence only radial deflection w is considered and this deflection only depends on time t and some azimuthal angle θ (see previous figure). The structure is inflated and the inner pressure is denoted $p_i(t)$; it is supposed to be only function of time but actually it should be constant due to the assumptions made for the body if no change of temperature is expected. There exists a set of functions w_n over which the deflection $w(x, y, t)$ can be projected :

$$w(x, y, t) = \sum_{n=1}^{\infty} q_n(t) w_n(\theta). \tag{2}$$

It measures the deflection from the undeformed body shape and q_n are the weights of each mode. The formulation of the problem is given for an arbitrary symmetric shape under the asumption that we are able to find an analytical expression for the modal deformations w_n. For particular shapes, such as cylindrical cylinder section or wedges and for suitable boundary conditions, expressions of the w_n are simple to obtain.

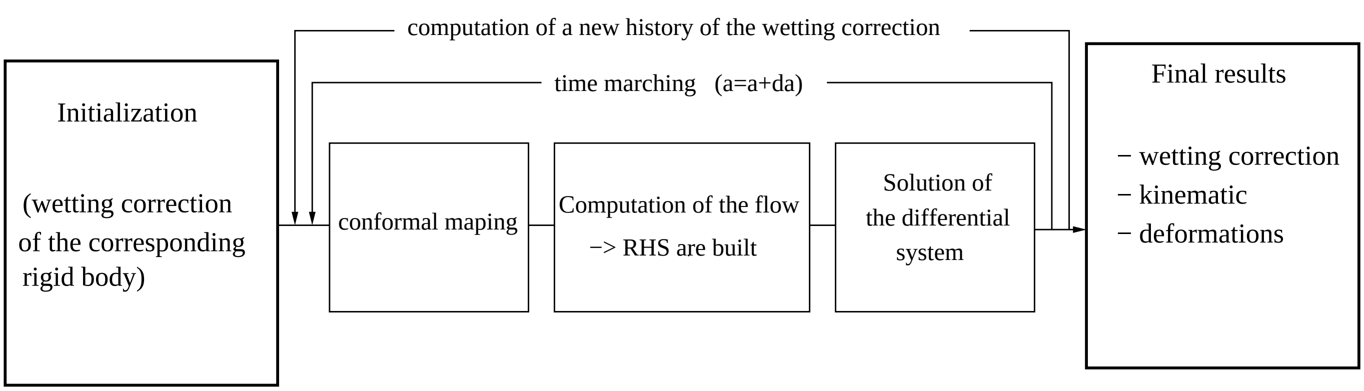
3) Hydrodynamic problem

The hydrodynamic problem is solved according to the assumptions of the so-called generalized Wagner model as introduced by Zhao *et al.* (1996). This means that the boundary condition is prescribed on the exact wetted surface but the free surface is linearized on lines emanating from the contact point. As the problem is symmetric, there is only one contact point to determine, say $a(t)$. The time integration of the kinematic free surface condition is thus used to provide us with $a(t)$:

$$f(a(t), t) - h(t) = \int_0^t \varphi_{,y}(a(t), \eta(a(\tau), \tau), \tau) d\tau. \tag{3}$$

It should be noted that now the problem becomes much more complicated than the rigid case since the shape of the body, denoted f, evolves locally in time. However we can assert that the vertical component of the contact point only depend on $a(t)$, if we admit that $a(t)$ increases monotonically in time.

As done by Mei *et al.* (1999) and further developed by Malleron *et al.* (2007), conformal mappings are used to transform the fluid domain into the half lower space. The hydrodynamic problem is then reformulated as a Riemann-Hilbert problem and a quasi-analytical solution can be exhibited. Through the successive conformal mappings used, the wetting surface becomes a unit circle (actually its half). The BVP (1) is highly nonlinear and an iterative procedure must be implemented to converge towards its solution (see figure below). At each iteration, the time evolution of the problem should be computed by iteration too. The



history of the weights q_n is built step by step by solving a differential system, detailed in the next section. For each value of $a(t)$, wich is considered as the evolution variable of the problem, the right hand side of the differential system should be explicited. This requires both the computation of the velocity potential and the knowledge of the expansion velocity of the wetting surface $\frac{da}{dt}$. This latter data is known from the previous computation of the wetting correction time history (first level of iteration). First we consider the impermeability condition which is reformulated as

$$\varphi_{,n} = -V\vec{y}\cdot\vec{n} + \sum_{n=1}^{\infty} \dot{q}_n w_n(\theta). \tag{4}$$

Knowing the shape at a given time, this allows to break down the potential φ into two components

$$\varphi(x,y,t) = V(t)\phi(x,y,t) + \sum_{n=1}^{\infty} \dot{q}_n \phi_n(x,y,t). \tag{5}$$

Correspondingly the vertical velocity on the free surface, which appears in the integrand of (3), will be broke down similarly. The body shape is fully described when $a(t)$ and the weights $q_n(t)$ are known. If so the components $\phi(x,y,t)$ and $\phi_n(x,y,t)$ are solutions of the following BVP respectively

$$\begin{cases} \Delta\phi = 0 & y<0 \\ \phi = 0 & \text{on the free surface} \\ \phi_{,n} = -\vec{y}\cdot\vec{n} & \text{on the wetted surface } D(t) \\ \phi \to 0 & (x^2+y^2)\to\infty, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\phi_n = 0 & y<0 \\ \phi_n = 0 & \text{on the free surface} \\ \phi_{n,n} = w_n(\theta) & \text{on the wetted surface } D(t) \\ \phi_n \to 0 & (x^2+y^2)\to\infty. \end{cases} \tag{6}$$

By using adequate conformal mappings, these BVPs are formulated as Riemann-Hilbert problems. We denote α the azimuth in the complex plane where the body contour is a unit circle. After some manipulations, we end up with the expressions of ϕ and ϕ_n on the body contour

$$\phi(\alpha) = -\sum_{m=1}^{\infty} A_m \sin(m\alpha), \quad \phi_n(\alpha) = -\sum_{m=1}^{\infty} C_{mn}\sin(m\alpha) \tag{7}$$

which are quite similar and where the computation of A_m and C_{mn} does not require a significant effort.

4) Coupled problem

The coupling is performed after some transformations of the time differential system for the deflection. Attention should be paid to the fact that in practice, the hydrodynamic problem is solved for an associated "double body problem" (see Mei *et al.*). The physical pressure is then given by Bernoulli law :

$$p(x,y,t) = -\rho_f\Phi_{,t} - \frac{1}{2}\rho_f(\vec{\nabla}\Phi)^2 = -\rho_f\frac{d\varphi}{dt} + \rho_f\frac{d}{dt}(Vy') + \rho_f\dot{\vec{X}}\vec{\nabla}\varphi - \frac{1}{2}\rho_f(\varphi_{,x}^2 + (\varphi_{,y}-V)^2), \tag{8}$$

where ρ_f is the density of the fluid and Φ the velocity potential in the "double body problem". $\dot{\vec{X}}$ is the local velocity along the body contour and y and y' are linked by : $y' = y - \eta(a(t))$. We collect the non linear terms that are denoted :

$$U(\theta,t) = \rho_f\dot{\vec{X}}\vec{\nabla}\varphi - \frac{1}{2}\rho_f(\varphi_{,x}^2 + (\varphi_{,y}-V)^2). \tag{9}$$

Introducing this in the PDE for w, we get

$$L(w) + \rho_s e(\ddot{w} - \dot{V}) = -\rho_f\frac{d(\varphi - Vy')}{dt} + U(\theta,t) - p_i(t). \tag{10}$$

We collect the time derivatives and we introduce the variable $Q(\theta,t)$ as

$$Q(\theta,t) = \rho_s e(\dot{w} - V) + \rho_f(\varphi - Vy'), \tag{11}$$

yielding the differential system

$$\dot{w} = \frac{Q}{\rho_s e} + V - \mu(\varphi - Vy'), \quad \mu = \frac{\rho_f}{\rho_s e}, \tag{12}$$

$$\dot{Q} = -L(w) + U(\theta, t) - p_i(t). \tag{13}$$

The variables w and Q are then projected on the normal modes and by using the orthogonality of these functions for the inner product

$$\int_0^{2\pi} w_m(\theta) w_n(\theta) d\theta = W_{mn}\delta_{mn}. \tag{14}$$

We end up with

$$\dot{q}_m = \frac{Q_m}{\rho_s e} - \frac{2\mu}{W_{mm}} \int_0^{\theta_a} (\varphi - Vy') w_m(\theta) d\theta, \tag{15}$$

and :

$$\dot{Q}_m = -L(q_m) + \frac{1}{W_{mm}} \int_0^{\theta_a} U(\theta, t) w_m(\theta) d\theta = R_Q, \tag{16}$$

where θ_a is the azimuthal coordinate of the contact point. The decomposition of φ in equation (5) is then introduced in (15) which is written in a matrix form

$$\sum_{p=1}^{\infty} S_{mp}\dot{q}_p = R_q = \frac{Q_m}{\rho_s e} - \frac{2\mu V}{W_{mm}} \int_0^{\theta_a} \phi w_m(\theta) d\theta, \quad S_{mp} = \delta_{mp} + \frac{2\mu}{W_{mm}} \int_0^{\theta_a} \phi_p w_m(\theta) d\theta. \tag{17}$$

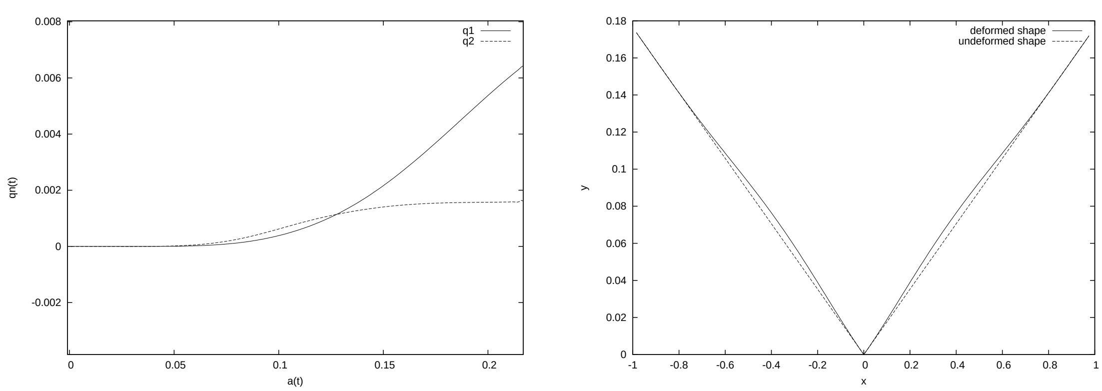
The equations (17) and (16) are solved in time to build the time history of $q_n(t)$ which is necessary to solve equation (3). These equations are written as a Cauchy problem in the form

$$\begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{dq}{da} \\ \frac{dQ}{da} \end{pmatrix} = \frac{1}{U(a)} \sum_{j=0}^{\infty} b_j a^j \begin{pmatrix} R_q \\ R_Q \end{pmatrix}. \tag{18}$$

If the vertical velocity is not set as constant, Newton law must be time integrated as well.

5) Preliminary results

The figure below shows the history of the two first mode's weight during the impact of an elastic wedge, clamped at its apex and its instantaneous deformation when $a(t) = 0.15$. In that case, the mode's shapes are given by : $w_n(s) = C_m \left(\cosh \frac{k_m s}{R} - \cos \frac{k_m s}{R}\right) - S_m \left(\sinh \frac{k_m s}{R} - \sin \frac{k_m s}{R}\right)$, where C_m and S_m are cofficients of nondimensionalization. The wedge is made of a material with the following characteristics : $E = 2.1.10^{11}\, Pa$, $\nu = 0.34$ and $\rho = 2700\, kg.m^{-3}$. Its thickness is $e = 0.01\, m$ and $R = 1\, m$.



More details about the computation of the different coefficients will be given in the final communication as well as more practical results.

6) References

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